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Deformed algebras, position-dependent effective masses and curved spaces: an exactly solvable Coulomb problem

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Abstract

We show that there exist some intimate connections between three unconventional Schrödinger equations based on the use of deformed canonical commutation relations, of a position-dependent effective mass or of a curved space. This occurs whenever a specific relation between the deforming function, the position-dependent mass and the (diagonal) metric tensor holds true. We illustrate these three equivalent approaches by considering a new Coulomb problem and solving it by means of supersymmetric quantum mechanical and shape invariance techniques. We show that in contrast with the conventional Coulomb problem, the new one gives rise to only a finite number of bound states.

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1. Introduction

In recent years there has been a growing interest in the study of quantum mechanical systems with a position-dependent effective mass due to their applications in condensed-matter physics. Initially proposed to describe impurities in crystals [1], the effective-mass theory has become an essential ingredient in the description of electronic properties of, for instance, semiconductors [2] and quantum dots [3].

The concept of effective mass is also relevant in connection with the energy-density functional approach to the quantum many-body problem. This formalism has been extensively used in nuclei [4], quantum liquids [5], ^{3}He clusters [6] and metal clusters [7].

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The study of quantum mechanical systems with position-dependent mass raises some important conceptual problems, such as the ordering ambiguity of the momentum and mass operators in the kinetic energy term, the boundary conditions at abrupt interfaces characterized by discontinuities in the mass function and the Galilean invariance of the theory (see, e.g., [8, 9]).

In the standard case of constant mass, exactly solvable (ES) Schrödinger equations have played an important role because they provide both a conceptual understanding of some physical phenomena and a testing ground for some approximation schemes. Many different approaches have been used including point canonical transformations (PCT) [10], Lie algebraic methods [11], supersymmetric quantum mechanical (SUSYQM) and shape invariance (SI) techniques [12]. The class of ES potentials has also been extended to the so-called quasi-exactly solvable (QES) potentials [13] and the conditionally exactly solvable (CES) ones [14]. During the last few years, some of these developments have been generalized to the case of position-dependent mass [15–22].

The main purpose of the present paper is to establish some connections between the Schrödinger equation with position-dependent mass and two other extensions of the conventional Schrödinger equation in current use.

The first one is based on the replacement of the standard commutation relations by deformed ones. A motivation for such an approach is the possibility of describing, for a special choice of deformation, nonzero minimal uncertainties in position and/or momentum [23]. This is in line with several investigations in string theory and quantum gravity, which suggest the existence of a finite lower bound to the possible resolution of length and/or momentum (see, e.g., [24]). In this context, several ES problems related to the harmonic oscillator have recently been considered using either the PCT [25] or the SUSYQM [26] method.

The second extension has to do with the Schrödinger equation in curved space. Interest in such a problem, which dates back to Schrödinger himself [27], has been revived with the advent of deformed algebras [28, 29]. Since then, many studies have been devoted to this topic, especially in the case of spaces of constant curvature (see, e.g., [30]), but also in the more general case of nonconstant curvature (see, e.g., [31]).

Another purpose of the present paper is to illustrate these three equivalent approaches by considering a new Coulomb problem with a definite dependence of the mass on the radial variable. We plan to show that the corresponding Schrödinger equation is ES by using the same kind of SUSYQM approach as that recently applied to the harmonic oscillator problem with nonzero minimal uncertainties in position and/or momentum [26].

Our paper is organized as follows. Connections between some unconventional Schrödinger equations are established in section 2. Section 3 deals with a new Coulomb problem. Finally, section 4 contains the conclusion.

2. Connections between some unconventional Schrödinger equations

In the present section, we will successively consider the cases where the Schrödinger equation is used in combination with some deformed canonical commutation relations or it contains a position-dependent mass or else the underlying space is curved and we will establish some connections between these three approaches. For definiteness sake, we will restrict ourselves here to the example of a single particle in three dimensions, but it should be clear that similar relations exist in more general cases.

2.1. Deformed canonical commutation relations

In the conventional canonical commutation relations

$$[x_i, x_j] = 0 \tag{2.1}$$

$$[x_i, p_j] = \mathrm{i}\hbar\delta_{i,j} \tag{2.2}$$

$$[p_i, p_j] = 0 \tag{2.3}$$

where $i = 1, 2, 3$, let us replace the momentum components $p_i = -\mathrm{i}\hbar\nabla_i = -\mathrm{i}\hbar\partial/\partial x_i$ by some deformed Hermitian operators

$$\pi_i = \sqrt{f(\boldsymbol{x})}\, p_i \sqrt{f(\boldsymbol{x})} \tag{2.4}$$

where the (positive real) deforming function f depends on the coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$. It is then straightforward to show that while equation (2.1) still holds, equations (2.2) and (2.3) are replaced by

$$[x_i, \pi_j] = \mathrm{i}\hbar f(\boldsymbol{x})\delta_{i,j} \tag{2.5}$$

$$[\pi_i, \pi_j] = -\mathrm{i}\hbar[f_i(\boldsymbol{x})\pi_j - f_j(\boldsymbol{x})\pi_i] \tag{2.6}$$

where $f_i(\boldsymbol{x}) \equiv \nabla_i f(\boldsymbol{x})$.

In the special case where f only depends on the radial variable $r = \left(\sum_i x_i^2\right)^{1/2}$, equations (2.5) and (2.6) become

$$[x_i, \pi_j] = \mathrm{i}\hbar f(r)\delta_{i,j} \tag{2.7}$$

$$[\pi_i, \pi_j] = -\mathrm{i}\hbar \frac{f(r)f'(r)}{r}\epsilon_{ijk}l_k \tag{2.8}$$

where $f'(r) = \mathrm{d}f(r)/\mathrm{d}r$, ϵ_{ijk} is the antisymmetric tensor and

$$l_i = \epsilon_{ijk}x_j p_k \tag{2.9}$$

are the angular momentum components. The last satisfy the usual commutation relations with x_i and π_i, i.e.,

$$[l_i, x_j] = \mathrm{i}\hbar\epsilon_{ijk}x_k \qquad [l_i, \pi_j] = \mathrm{i}\hbar\epsilon_{ijk}\pi_k \tag{2.10}$$

from which it follows that the relation $[l_i, f(r)] = 0$ holds true. Such a property has actually been used in the derivation of (2.8).

Let us now consider a deformed Schrödinger operator

$$H_1 = \frac{1}{2m_0}\boldsymbol{\pi}^2 + V_1(\boldsymbol{x}) \tag{2.11}$$

where m_0 denotes the (constant) mass and $V_1(\boldsymbol{x})$ is some potential. The corresponding deformed Schrödinger equation reads

$$\left[-\frac{\hbar^2}{2m_0}(\sqrt{f(\boldsymbol{x})}\boldsymbol{\nabla}\sqrt{f(\boldsymbol{x})})^2 + V_1(\boldsymbol{x})\right]\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x}). \tag{2.12}$$

The problem can be reformulated in terms of dimensionless operators $\boldsymbol{X} = \boldsymbol{x}/a$, $\boldsymbol{P} = a\boldsymbol{p}/\hbar$, $\boldsymbol{\Pi} = a\boldsymbol{\pi}/\hbar$, where a denotes some characteristic length. Such new operators satisfy commutation relations similar to (2.1), (2.2) and (2.3) (or (2.1), (2.5) and (2.6)) where $\hbar$ is set equal to 1 and l_i is replaced by $L_i = \epsilon_{ijk}X_j P_k$. A dimensionless Hamiltonian can be defined by

$$h_1 = \frac{H_1}{V_0} = \frac{1}{2}\boldsymbol{\Pi}^2 + U_1(\boldsymbol{X}) \tag{2.13}$$

where $V_1(\boldsymbol{x}) = V_0 U_1(\boldsymbol{X})$ and $V_0 = \hbar^2/(m_0 a^2)$. Equation (2.12) then becomes

$$\left[-\tfrac{1}{2}(\sqrt{f(\boldsymbol{X})}\nabla\sqrt{f(\boldsymbol{X})})^2 + U_1(\boldsymbol{X})\right]\psi(\boldsymbol{X}) = e\psi(\boldsymbol{X}) \tag{2.14}$$

where $e = E/V_0$ and for the sake of simplicity, we keep the notation ∇_i to denote the derivatives $\partial/\partial X_i$ with respect to the dimensionless variables X_i.

2.2. Position-dependent mass

When the mass $m(\boldsymbol{x})$ is position dependent, it no longer commutes with the momentum $\boldsymbol{p} = -\mathrm{i}\hbar\nabla$, so that there are many ways of generalizing the usual form of the kinetic energy $(2m_0)^{-1}\boldsymbol{p}^2$, valid for a constant mass m_0, in order to obtain a Hermitian operator. This ordering ambiguity has been most debated (see, e.g., [8, 9]). Here we will not make any specific choice and will therefore adopt the general two-parameter form of the kinetic energy term, as originally proposed by von Roos [32].

Hence we consider a Hamiltonian H_2 defined by

$$H_2 = -\frac{\hbar^2}{4}[m^{\delta'}(\boldsymbol{x})\nabla m^{\kappa'}(\boldsymbol{x})\nabla m^{\lambda'}(\boldsymbol{x}) + m^{\lambda'}(\boldsymbol{x})\nabla m^{\kappa'}(\boldsymbol{x})\nabla m^{\delta'}(\boldsymbol{x})] + V_2(\boldsymbol{x}) \tag{2.15}$$

where $V_2(\boldsymbol{x})$ is some potential and the parameters δ', κ', λ' are constrained by the condition $\delta' + \kappa' + \lambda' = -1$. On expressing the position-dependent mass $m(\boldsymbol{x})$ as

$$m(\boldsymbol{x}) = m_0 M(\boldsymbol{x}) \qquad M(\boldsymbol{x}) = \frac{1}{f^2(\boldsymbol{x})} \tag{2.16}$$

where m_0 is a constant mass and $M(\boldsymbol{x})$ is a dimensionless position-dependent mass, equation (2.15) can be rewritten as

$$H_2 = -\frac{\hbar^2}{4m_0}[f^{\delta}(\boldsymbol{x})\nabla f^{\kappa}(\boldsymbol{x})\nabla f^{\lambda}(\boldsymbol{x}) + f^{\lambda}(\boldsymbol{x})\nabla f^{\kappa}(\boldsymbol{x})\nabla f^{\delta}(\boldsymbol{x})] + V_2(\boldsymbol{x}) \tag{2.17}$$

with $\delta + \kappa + \lambda = 2$. For the special choice $\delta' = \lambda' = 0$ and $\kappa' = -1$ or $\delta = \lambda = 0$ and $\kappa = 2$, equations (2.15) and (2.17) reduce to the most common BenDaniel–Duke form [33]

$$H_2' = -\frac{\hbar^2}{2}\nabla\frac{1}{m(\boldsymbol{x})}\nabla + V_2(\boldsymbol{x}) = -\frac{\hbar^2}{2m_0}\nabla f^2(\boldsymbol{x})\nabla + V_2(\boldsymbol{x}). \tag{2.18}$$

We now plan to show that the Hamiltonian (2.17) can be transformed into

$$H_2 = -\frac{\hbar^2}{2m_0}\sqrt{f(\boldsymbol{x})}\nabla f(\boldsymbol{x})\nabla\sqrt{f(\boldsymbol{x})} + V_{2,\mathrm{eff}}(\boldsymbol{x}) \tag{2.19}$$

for some appropriate choice of the effective potential $V_{2,\mathrm{eff}}(\boldsymbol{x})$. Since

$$\begin{aligned} f^{\delta}\nabla f^{\kappa}\nabla f^{\lambda} &= f^{\delta}\nabla f^{\frac{1}{2}-\delta} f f^{\frac{1}{2}-\lambda}\nabla f^{\lambda} \\ &= \left[f^{1/2}\nabla + \left(\tfrac{1}{2}-\delta\right) f^{-1/2}\boldsymbol{f}\right] f\left[\nabla f^{1/2} - \left(\tfrac{1}{2}-\lambda\right) f^{-1/2}\boldsymbol{f}\right] \\ &= \sqrt{f}\nabla f\nabla\sqrt{f} + (\lambda-\delta)\sqrt{f}\,\boldsymbol{f}\cdot\nabla\sqrt{f} - \left(\tfrac{1}{2}-\lambda\right) f\,\mathrm{div}\,\boldsymbol{f} \\ &\quad - \left(\tfrac{1}{2}-\delta\right)\left(\tfrac{1}{2}-\lambda\right)\boldsymbol{f}^2 \end{aligned} \tag{2.20}$$

with $\boldsymbol{f} = \nabla f$, we indeed obtain

$$\begin{aligned} &f^{\delta}\nabla f^{\kappa}\nabla f^{\lambda} + f^{\lambda}\nabla f^{\kappa}\nabla f^{\delta} \\ &\qquad = 2\sqrt{f}\nabla f\nabla\sqrt{f} - (1-\delta-\lambda) f\,\mathrm{div}\,\boldsymbol{f} - 2\left(\tfrac{1}{2}-\delta\right)\left(\tfrac{1}{2}-\lambda\right)\boldsymbol{f}^2. \end{aligned} \tag{2.21}$$

We conclude that equation (2.19) is valid for

$$V_{2,\mathrm{eff}}(\boldsymbol{x}) = V_2(\boldsymbol{x}) + \frac{\hbar^2}{2m_0}\left[\frac{1}{2}(1-\delta-\lambda)f(\boldsymbol{x})\,\mathrm{div}\,\boldsymbol{f}(\boldsymbol{x}) + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)\boldsymbol{f}^2(\boldsymbol{x})\right]. \tag{2.22}$$

On taking equation (2.4) into account, it becomes clear that the Hamiltonian H_2 with position-dependent mass (2.16) may be considered as a deformed Schrödinger Hamiltonian H_1, as given in (2.11), with $V_1(\boldsymbol{x}) = V_{2,\mathrm{eff}}(\boldsymbol{x})$.

In terms of dimensionless variables $\boldsymbol{X} = \boldsymbol{x}/a$, there corresponds to H_2 a dimensionless Hamiltonian

$$h_2 = \frac{H_2}{V_0} = -\frac{1}{2}\sqrt{f(\boldsymbol{X})}\nabla f(\boldsymbol{X})\nabla\sqrt{f(\boldsymbol{X})} + U_{2,\mathrm{eff}}(\boldsymbol{X}) \tag{2.23}$$

and a corresponding Schrödinger equation

$$\left[-\tfrac{1}{2}(\sqrt{f(\boldsymbol{X})}\nabla\sqrt{f(\boldsymbol{X})})^2 + U_{2,\mathrm{eff}}(\boldsymbol{X})\right]\psi(\boldsymbol{X}) = e\psi(\boldsymbol{X}) \tag{2.24}$$

where $U_{2,\mathrm{eff}}(\boldsymbol{X}) = V_{2,\mathrm{eff}}(\boldsymbol{x})/V_0$, $e = E/V_0$ and $V_0 = \hbar^2/(m_0a^2)$. We conclude that solving the deformed Schrödinger equation (2.14) will also provide us with the solutions of equation (2.24) provided

$$U_{2,\mathrm{eff}}(\boldsymbol{X}) = U_1(\boldsymbol{X}). \tag{2.25}$$

In the special case of a central potential $V(r)$ with a mass depending only on the radial variable r, i.e., $m(r) = m_0M(r) = m_0/f^2(r)$, the effective potential (2.22) reduces to

$$V_{2,\mathrm{eff}}(r) = V_2(r) + \frac{\hbar^2}{2m_0}\left\{\frac{1}{2}(1-\delta-\lambda)f(r)\left[\frac{2}{r}f'(r) + f''(r)\right] + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)f'^2(r)\right\}. \tag{2.26}$$

In terms of dimensionless quantities, this becomes

$$U_{2,\mathrm{eff}}(\rho) = U_2(\rho) + \frac{1}{2}\left\{\frac{1}{2}(1-\delta-\lambda)f(\rho)\left[\frac{2}{\rho}f'(\rho) + f''(\rho)\right] + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)f'^2(\rho)\right\} \tag{2.27}$$

for a mass $m(\rho) = m_0M(\rho) = m_0/f^2(\rho)$, where $\rho = r/a$.

2.3. Curved space

Let us consider a curved space, whose squared line element is given by

$$\mathrm{d}s^2 = g_{ij}(\boldsymbol{x})\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.28}$$

where the metric tensor is assumed to be of the form

$$g_{ij}(\boldsymbol{x}) = g_{ii}(\boldsymbol{x})\delta_{i,j} = D_i^2(\boldsymbol{x})\delta_{i,j} \tag{2.29}$$

with $g_{ii}(\boldsymbol{x}) = D_i^2(\boldsymbol{x})$ independent of i, i.e.,

$$g_{ii}(\boldsymbol{x}) = g(\boldsymbol{x}) \qquad D_i(\boldsymbol{x}) = D(\boldsymbol{x}). \tag{2.30}$$

The corresponding Laplacian operator [34]

$$\Delta = \frac{1}{D_1D_2D_3}\left(\frac{\partial}{\partial x_1}\frac{D_2D_3}{D_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{D_1D_3}{D_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{D_1D_2}{D_3}\frac{\partial}{\partial x_3}\right) \tag{2.31}$$

then reduces to

$$\Delta = \frac{1}{D^3(\boldsymbol{x})}\boldsymbol{\nabla} D(\boldsymbol{x})\boldsymbol{\nabla} = \boldsymbol{\nabla}\frac{1}{D^2(\boldsymbol{x})}\boldsymbol{\nabla} + \frac{3}{D^3(\boldsymbol{x})}\boldsymbol{D}(\boldsymbol{x})\cdot\boldsymbol{\nabla} \tag{2.32}$$

where $\nabla_i = \partial/\partial x_i$ and $\boldsymbol{D}(\boldsymbol{x}) = \boldsymbol{\nabla} D(\boldsymbol{x})$. On expressing $g(\boldsymbol{x})$ and $D(\boldsymbol{x})$ as

$$g(\boldsymbol{x}) = \frac{1}{f^2(\boldsymbol{x})} \qquad D(\boldsymbol{x}) = \frac{1}{f(\boldsymbol{x})} \tag{2.33}$$

Δ can be rewritten as

$$\Delta = \boldsymbol{\nabla} f^2(\boldsymbol{x})\boldsymbol{\nabla} - 3f(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{\nabla}. \tag{2.34}$$

In such a curved space, let us now consider some Hamiltonian

$$H_3 = -\frac{\hbar^2}{2m_0}\Delta + V_3(\boldsymbol{x}) \tag{2.35}$$

where m_0 denotes a (constant) mass and $V_3(\boldsymbol{x})$ is some potential. There corresponds to it a dimensionless Hamiltonian $h_3 = H_3/V_0$, depending on a dimensionless potential $U_3(\boldsymbol{X}) = V_3(\boldsymbol{x})/V_0$, where $\boldsymbol{X} = \boldsymbol{x}/a$ and $V_0 = \hbar^2/(m_0a^2)$.

From equation (2.34), it follows that the corresponding Schrödinger equation reads

$$\left[-\tfrac{1}{2}\boldsymbol{\nabla} f^2(\boldsymbol{X})\boldsymbol{\nabla} + \tfrac{3}{2}f(\boldsymbol{X})\boldsymbol{f}(\boldsymbol{X})\cdot\boldsymbol{\nabla} + U_3(\boldsymbol{X})\right]\tilde{\psi}(\boldsymbol{X}) = e\tilde{\psi}(\boldsymbol{X}) \tag{2.36}$$

where here $\nabla_i = \partial/\partial X_i$. The second term on the left-hand side containing first-order derivatives can be eliminated by setting

$$\tilde{\psi}(\boldsymbol{X}) = [f(\boldsymbol{X})]^{3/2}\psi(\boldsymbol{X}). \tag{2.37}$$

The resulting equation can be written as

$$\left\{-\tfrac{1}{2}\boldsymbol{\nabla} f^2(\boldsymbol{X})\boldsymbol{\nabla} + \tfrac{3}{4}\left[\tfrac{1}{2}\boldsymbol{f}^2(\boldsymbol{X}) - f(\boldsymbol{X})\,\mathrm{div}\,\boldsymbol{f}(\boldsymbol{X})\right] + U_3(\boldsymbol{X})\right\}\psi(\boldsymbol{X}) = e\psi(\boldsymbol{X}) \tag{2.38}$$

or, alternatively,

$$\left[-\tfrac{1}{2}(\sqrt{f(\boldsymbol{X})}\boldsymbol{\nabla}\sqrt{f(\boldsymbol{X})})^2 + U_{3,\mathrm{eff}}(\boldsymbol{X})\right]\psi(\boldsymbol{X}) = e\psi(\boldsymbol{X}) \tag{2.39}$$

where

$$U_{3,\mathrm{eff}}(\boldsymbol{X}) = U_3(\boldsymbol{X}) - \tfrac{1}{2}f(\boldsymbol{X})\,\mathrm{div}\,\boldsymbol{f}(\boldsymbol{X}) + \tfrac{1}{2}\boldsymbol{f}^2(\boldsymbol{X}). \tag{2.40}$$

Note that in the last step, we used equation (2.20) with $\delta = \lambda = 0$ and $\kappa = 2$.

We have therefore proved that solving the deformed Schrödinger equation (2.14) also provides us with the solutions of equation (2.36) provided we make the identification

$$U_{3,\mathrm{eff}}(\boldsymbol{X}) = U_1(\boldsymbol{X}) \tag{2.41}$$

and take equations (2.37) and (2.40) into account.

Note that whenever the potential U_3 and the metric tensor $g = 1/f^2$ only depend on $\rho = \left(\sum_i X_i^2\right)^{1/2}$, the effective potential reduces to

$$U_{3,\mathrm{eff}}(\rho) = U(\rho) - \frac{1}{2}f(\rho)\left(\frac{2}{\rho}f'(\rho) + f''(\rho)\right) + \frac{1}{2}f'^2(\rho). \tag{2.42}$$

As shown in the appendix, in such a special case, the space curvature takes a simple form in terms of the metric $g(\rho)$ or the deforming function $f(\rho)$,

$$R = 2\left(-\frac{4}{\rho}f(\rho)f'(\rho) - 2f(\rho)f''(\rho) + 3f'^2(\rho)\right). \tag{2.43}$$

From this equation, it is clear that for a generic choice of $f(\rho)$, R is not a constant, but some function of ρ.

We can summarize the results of this section as follows: we have established some closed links between the deformed Schrödinger equation, the Schrödinger equation with position-dependent mass and the Schrödinger equation in curved space whenever the deforming function $f(\boldsymbol{x})$, the (dimensionless) position-dependent mass $M(\boldsymbol{x})$ and the (diagonal) metric $g(\boldsymbol{x})$ are connected through the relations

$$f^2(\boldsymbol{x}) = \frac{1}{M(\boldsymbol{x})} = \frac{1}{g(\boldsymbol{x})}. \tag{2.44}$$

3. An exactly solvable Coulomb problem

Let us illustrate the general results obtained in section 2 by considering a deforming function and a potential energy depending only on r and given by

$$f(r) = 1 + \overline{\alpha} r \qquad V(r) = -\frac{Ze^2}{r} \tag{3.1}$$

respectively. Here $\overline{\alpha}$ is some non-negative parameter, Z is the atomic number and e is the electronic charge. For the Coulomb potential, the characteristic length is the Bohr radius $a = \hbar^2/(m_0 e^2)$, where m_0 denotes the (undressed) mass of the electron. It follows that $V_0 = \hbar^2/(m_0 a^2) = m_0 e^4/\hbar^2 = 2R$, where R is the Rydberg constant.

In terms of the dimensionless radial variable $\rho = r/a$, equation (3.1) can be rewritten as

$$f(\rho) = 1 + \alpha\rho \qquad U(\rho) = -\frac{Z}{\rho} \tag{3.2}$$

where $\alpha = a\overline{\alpha}$.

3.1. Deformed canonical commutation relations

For the choice made in equation (3.2), the deformed canonical commutation relations satisfied by the dimensionless operators X_i, Π_i read

$$[X_i, X_j] = 0 \tag{3.3}$$

$$[X_i, \Pi_j] = \mathrm{i}(1 + \alpha\rho)\delta_{i,j} \tag{3.4}$$

$$[\Pi_i, \Pi_j] = -\mathrm{i}(1 + \alpha\rho)\frac{\alpha}{\rho}\epsilon_{ijk}L_k \tag{3.5}$$

while, in spherical coordinates ρ, θ, φ, the deformed Schrödinger equation (2.14) becomes

$$\left\{-\frac{1}{2}\sqrt{f(\rho)}\left[f(\rho)\left(\frac{\partial^2}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial}{\partial\rho} - \frac{\boldsymbol{L}^2}{\rho^2}\right) + f'(\rho)\frac{\partial}{\partial\rho}\right]\sqrt{f(\rho)} - \frac{Z}{\rho}\right\}\psi(\rho,\theta,\varphi)$$
$$= e\psi(\rho,\theta,\varphi) \tag{3.6}$$

where $\boldsymbol{L}^2$ denotes the square of the angular momentum operator. In deriving equation (3.6), we have used the relation

$$\boldsymbol{\nabla} f(\rho)\boldsymbol{\nabla} = f(\rho)\boldsymbol{\nabla}^2 + \frac{f'(\rho)}{\rho}\boldsymbol{X}\cdot\boldsymbol{\nabla}. \tag{3.7}$$

As in the usual $f(\rho) = 1$ case, equation (3.6) is separable. On setting

$$\psi_{klm}(\rho,\theta,\varphi) = \frac{1}{\rho}R_{kl}(\rho)Y_{lm}(\theta,\varphi) \tag{3.8}$$

where $Y_{lm}(\theta,\varphi)$ is a spherical harmonics, we get the radial differential equation

$$h_1^{(l)}R_{kl}(\rho) = e_{kl}R_{kl}(\rho). \tag{3.9}$$

Here

$$h_1^{(l)} = -\frac{1}{2}\left(\sqrt{f(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\sqrt{f(\rho)}\right)^2 + U_1^{(l)}(\rho) \tag{3.10}$$

$$U_1^{(l)}(\rho) = \frac{1}{2}\left(f^2(\rho)\frac{L^2}{\rho^2} + \frac{f(\rho)f'(\rho)}{\rho}\right) - \frac{Z}{\rho} = \frac{1}{2}\left(-\frac{2Z-\alpha(2L^2+1)}{\rho} + \frac{L^2}{\rho^2} + \alpha^2(L^2+1)\right) \tag{3.11}$$

where

$$L^2 = l(l+1) \tag{3.12}$$

is the eigenvalue of $\boldsymbol{L}^2$ and k denotes the radial quantum number.

We now plan to show that equation (3.9) is ES by applying the same kind of SUSYQM methods as we used in [26] to solve the harmonic oscillator problem with nonzero minimal uncertainties in position and/or momentum.

To start with, the radial Hamiltonian $h_1^{(l)}$ can be factorized as

$$h_1^{(l)} = B^+(g,s)B^-(g,s) + \epsilon_0 \tag{3.13}$$

where the first-order operators $B^{\pm}(g,s)$ and the factorization energy ϵ_0 are given by

$$B^{\pm}(g,s) = \frac{1}{\sqrt{2}}\left(\mp\sqrt{f(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\sqrt{f(\rho)} - \frac{s}{\rho} + g\right) \tag{3.14}$$

$$s = l+1 \qquad g = \frac{Z - \frac{\alpha}{2}[(l+1)^2 + L^2 + 1]}{l+1} \tag{3.15}$$

$$\epsilon_0 = -\frac{1}{2}g^2 + \frac{1}{2}\alpha^2(L^2+1). \tag{3.16}$$

In the $\alpha \to 0$ limit, we get $s = l+1, g = Z/(l+1)$ and $\epsilon_0 = -Z^2/[2(l+1)^2]$, which correspond to the usual factorization for the Coulomb potential in conventional quantum mechanics [12].

Let us next consider a hierarchy of Hamiltonians

$$h_{1i}^{(l)} = B^+(g_i,s_i)B^-(g_i,s_i) + \sum_{j=0}^{i}\epsilon_j \qquad i = 0,1,2,\ldots \tag{3.17}$$

whose first member $h_{10}^{(l)}$ coincides with $h_1^{(l)}$ (hence $g_0 = g$ and $s_0 = s$), and let us impose a SI condition [12]

$$B^-(g_i,s_i)B^+(g_i,s_i) = B^+(g_{i+1},s_{i+1})B^-(g_{i+1},s_{i+1}) + \epsilon_{i+1}. \tag{3.18}$$

It can be easily shown that equation (3.18) is satisfied provided

$$s_i = s + i = l + i + 1 \tag{3.19}$$

$$g_i = \frac{g(l+1) - \frac{\alpha}{2}[l+1+(2l+3)i+i^2]}{l+i+1} + \frac{\alpha}{2} = \frac{Z - \frac{\alpha}{2}[(l+i+1)^2 + L^2 + 1]}{l+i+1} \tag{3.20}$$

$$\epsilon_i = \frac{1}{2}g_{i-1}^2 - \frac{1}{2}g_i^2 \tag{3.21}$$

for $i = 1, 2, \ldots$. Again in the $\alpha \to 0$ limit, we get back the usual result $s_i = l + i + 1$ and $g_i = Z/(l + i + 1)$.

The energy eigenvalues in equation (3.9) can now be obtained from equations (3.16) and (3.21) as

$$e_{kl} = e_k(g, s) = \sum_{i=0}^{k} \epsilon_i = -\frac{1}{2} g_k^2 + \frac{1}{2}\alpha^2(L^2 + 1). \tag{3.22}$$

Inserting equation (3.20) in equation (3.22) converts the latter into

$$e_{kl} = -\frac{1}{2}\left\{\frac{Z - \frac{\alpha}{2}[(l + k + 1)^2 + L^2 + 1]}{l + k + 1}\right\}^2 + \frac{1}{2}\alpha^2(L^2 + 1) \tag{3.23}$$

where L^2 is given in (3.12). In terms of the principal quantum number $n = k + l + 1$, the eigenvalues can be rewritten as

$$e_{nl} = -\frac{\left[Z - \frac{\alpha}{2}(L^2 + 1)\right]^2}{2n^2} - \frac{\alpha^2}{8}n^2 + \frac{\alpha}{2}\left[Z + \frac{\alpha}{2}(L^2 + 1)\right] \tag{3.24}$$

which, in the $\alpha \to 0$ limit, leads to the usual result $e_{nl} = -Z^2/(2n^2)$. Note that for $\alpha \neq 0$, there is an additional quadratic term in n, as well as an additional dependence on L^2.

This purely algebraic determination of the spectrum has now to be completed by a construction of the corresponding radial wavefunctions $R_{kl}(\rho)$, which should be normalizable on $(0, \infty)$ according to[4]

$$\int_0^\infty \mathrm{d}\rho |R_{kl}(\rho)|^2 = 1. \tag{3.25}$$

As we now plan to show, this restricts the allowed (integer) values of l and k in contrast with the standard Coulomb problem for which they may take any value in $\mathbb{N}$.

Let us first consider the ground state wavefunction $R_{0l}(\rho) = R_0(g, s; \rho)$ of $h_1^{(l)}$, which is a solution of the first-order differential equation

$$B^-(g, s)R_0(g, s; \rho) = 0. \tag{3.26}$$

It is given by

$$R_0(g, s; \rho) = \mathcal{N}_0(g, s)\rho^s(1 + \alpha\rho)^{-\left(\frac{g}{\alpha} + s + \frac{1}{2}\right)} \tag{3.27}$$

which is a square-integrable function provided

$$s > 0 \qquad g > 0. \tag{3.28}$$

In such a case, the normalization constant $\mathcal{N}_0(g, s)$ is

$$\mathcal{N}_0(g, s) = \left(\frac{\Gamma\left(2\frac{g}{\alpha} + 2s + 1\right)}{\Gamma\left(2\frac{g}{\alpha}\right)\Gamma(2s + 1)}\alpha^{2s+1}\right)^{1/2}. \tag{3.29}$$

From equation (3.15), it follows that the first inequality in (3.28) is automatically satisfied. However, the second one implies that l may only vary in the range $l = 0, 1, \ldots, l_{\max}$, where $l_{\max}$ is the largest integer fulfilling the condition

$$(l + 1)(2l + 1) < \frac{2Z}{\alpha} - 1. \tag{3.30}$$

[4] It should be noted that in contrast with [26], the scalar product is not changed by the deformation. With respect to (3.25), the properties $[B^+(g, s)]^\dagger = B^-(g, s)$ and $(h_1^{(l)})^\dagger = h_1^{(l)}$ hold true.

Let us next consider the excited state wavefunctions $R_{kl}(\rho) = R_k(g, s; \rho)$, $k = 1, 2, \ldots$, which can be determined from the recursion relation

$$R_{k+1}(g, s; \rho) = [e_{k+1}(g, s) - e_0(g, s)]^{-1/2} B^+(g, s) R_k(g_1, s_1; \rho) \tag{3.31}$$

where s_1 and g_1 are defined according to equations (3.19) and (3.20), respectively. It can be easily shown that the normalizable solutions of equation (3.31) are given by

$$R_k(g, s; \rho) = \mathcal{N}_k(g, s) P_k(g, s; \rho) \rho^s (1 + \alpha\rho)^{-\left(\frac{g_k}{\alpha} + s_k + \frac{1}{2}\right)} \tag{3.32}$$

where

$$s_k > 0 \qquad g_k > 0 \tag{3.33}$$

$P_k(g, s; \rho)$ denotes some kth-degree polynomial in ρ, satisfying the relation

$$P_{k+1}(g, s; \rho) = -\rho f(\rho) P_k'(g_1, s_1; \rho) + [-(2s + 1) + (g_{k+1} + g + k\alpha)\rho] P_k(g_1, s_1; \rho) \tag{3.34}$$

with $P_0(g, s; \rho) \equiv 1$, and $\mathcal{N}_k(g, s)$ is some normalization coefficient fulfilling the recursion relation

$$\begin{aligned} \mathcal{N}_{k+1}(g, s) &= \{2[e_{k+1}(g, s) - e_0(g, s)]\}^{-1/2} \mathcal{N}_k(g_1, s_1) \\ &= (s + k + 1) \left\{ (k + 1)(2s + k + 1) \left[g + \frac{\alpha}{2}(2s + k + 1) \right] \left[g - \frac{\alpha}{2}(k + 1) \right] \right\}^{-1/2} \\ &\quad \times \mathcal{N}_k(g_1, s_1). \end{aligned} \tag{3.35}$$

The second condition in (3.33) is equivalent to the inequality

$$(l + k + 1)^2 + l(l + 1) < \frac{2Z}{\alpha} - 1 \tag{3.36}$$

generalizing equation (3.30). It implies that both l and k run over some finite sets, $l = 0, 1, \ldots, l_{\max}$ and $k = 0, 1, \ldots, k_{\max}$. We therefore conclude that in contrast with the conventional Coulomb problem, the deformed one corresponding to $f(\rho) = 1 + \alpha\rho$ has only a finite number of bound states.

For the first few k values, explicit expressions of the polynomials $P_k(g, s; \rho)$ can be obtained by solving equation (3.34). For the first two excited states, for instance, we get

$$P_1(g, s; \rho) = -(s + s_1) + (g + g_1)\rho \tag{3.37}$$

$$\begin{aligned} P_2(g, s; \rho) &= (s + s_1)(s_1 + s_2) - [(s + s_2)(g_1 + g_2) + (s_1 + s_2)(g + g_2 + \alpha)]\rho \\ &\quad + (g + g_2)(g_1 + g_2)\rho^2. \end{aligned} \tag{3.38}$$

In the $\alpha \to 0$ limit, the radial wavefunctions obtained in the present section should give back the conventional ones [35][5]. Recalling that $s = l + 1$ and $g = Z/(l + 1)$ in such a limit, we easily get the usual result for the ground state wavefunction of $h_1^{(l)}$,

$$R_{0l}(\rho) = N_{0l} \rho^{l+1} \exp\left(-\frac{Z\rho}{l + 1} \right) \tag{3.39}$$

where

$$N_{0l} = \frac{1}{\sqrt{(2l + 2)!}} \left(\frac{2Z}{l + 1} \right)^{l+3/2}. \tag{3.40}$$

For the excited state wavefunctions, it can be shown that for $\alpha \to 0$, the polynomials $P_k(g, s; \rho)$ become

$$P_{kl}(\rho) = a_{kl} L_k^{(2l+1)}(t) \qquad t \equiv \frac{2Z\rho}{n} = \frac{2Z\rho}{k + l + 1} \tag{3.41}$$

[5] Contrary to what was done in [35], we use the conventional definition [36] of generalized Laguerre polynomials in equations (3.41), (3.43) and (3.44).

where

$$a_{kl} = (-1)^k \frac{k!(2l)!!(2k+2l+1)!}{(2k+2l)!!(k+2l+1)!} \tag{3.42}$$

and $L_k^{(2l+1)}(t)$ is a generalized Laguerre polynomial. Inserting (3.41) in equation (3.34) where α is set equal to zero, we indeed get the relation

$$(2l+2)t\frac{\mathrm{d}}{\mathrm{d}t}L_{k-1}^{(2l+3)}(t) = k(k+2l+2)L_k^{(2l+1)}(t) + [-(2l+2)(2l+3) + (k+2l+2)t]L_{k-1}^{(2l+3)}(t) \tag{3.43}$$

which can be proved to hold true from the known relations satisfied by generalized Laguerre polynomials [36]. Finally, on using equations (3.35), (3.40), (3.41) and (3.42), we obtain that the functions (3.32) lead to

$$R_{kl}(\rho) = N_{kl}\rho^{l+1}L_k^{(2l+1)}\left(\frac{2Z\rho}{n}\right)\exp\left(-\frac{Z\rho}{n}\right) \tag{3.44}$$

where

$$N_{kl} = (-1)^{n-l-1}\left(\frac{2Z}{n}\frac{(n-l-1)!}{2n(n+l)!}\right)^{1/2}\left(\frac{2Z}{n}\right)^{l+1}. \tag{3.45}$$

3.2. Position-dependent mass

Let us now make the choice (3.1) with $V_2(r) = V(r)$ in equation (2.15). The Hamiltonian H_2 then describes an electron in a Coulomb potential with a position-dependent mass

$$m(r) = \frac{m_0}{(1+\overline{\alpha}r)^2} \qquad \text{or} \qquad m(\rho) = \frac{m_0}{(1+\alpha\rho)^2} \tag{3.46}$$

decreasing from m_0 to 0 when the radial variable increases from 0 to ∞.

In the associated Schrödinger equation (2.24), the effective potential $U_{2,\mathrm{eff}}(\rho)$ reads

$$U_{2,\mathrm{eff}}(\rho) = -\frac{Z^*}{\rho} + \frac{1}{2}\alpha^2\left[1-\delta-\lambda+\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)\right] \tag{3.47}$$

$$Z^* \equiv Z - \frac{\alpha}{2}(1-\delta-\lambda). \tag{3.48}$$

Apart from some additive constant, it amounts to a Coulomb potential depending on an effective charge Z^*. Hence we can avail ourselves of the results proved in sections 2.2 and 3.1 to provide the solutions e_{kl} (or e_{nl}) and $\psi_{klm}(\rho,\theta,\varphi)$ of equation (2.24).

The spectrum is given by

$$e_{nl} = -\frac{\left[Z-\frac{\alpha}{2}(L^2+2-\delta-\lambda)\right]^2}{2n^2} - \frac{\alpha^2}{8}n^2 + \frac{\alpha}{2}\left[Z+\frac{\alpha}{2}(L^2+\delta+\lambda)\right] + \frac{\alpha^2}{2}\left[1-\delta-\lambda+\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)\right] \tag{3.49}$$

where $n = k+l+1$. From (3.36) and (3.48), it results that the range of allowed l and k values is now determined by the modified condition

$$(l+k+1)^2 + l(l+1) < \frac{2Z}{\alpha} - (2-\delta-\lambda) \tag{3.50}$$

which, apart from Z, depends on the mass parameter α and on the parameter $\kappa = 2-\delta-\lambda$ related to the mass-ordering ambiguity problem. The distances between consecutive levels are also entirely governed by these two parameters.

Finally, the wavefunctions $\psi_{klm}(\rho,\theta,\varphi)$ can be obtained from equations (3.8) and (3.32) with Z^* substituted for Z.

3.3. Curved space

Let us finally make the choice (3.1) with $V_3(r) = V(r)$ in equation (2.35). For the metric tensor $g(\rho) = 1/f^2(\rho) = 1/(1+\alpha\rho)^2$, we obtain from (2.43) that the nonconstant space curvature is given by

$$R = -2\alpha\left(\frac{4}{\rho} + \alpha\right) \tag{3.51}$$

and is therefore negative for all ρ values. In such a space, $V_3(r)$, as given in (3.1), may not be interpreted as a Coulomb potential since, as shown in the appendix, the latter assumes there a more complicated form.

In the associated Schrödinger equation (2.39), the effective potential $U_{3,\mathrm{eff}}(\rho)$ reads

$$U_{3,\mathrm{eff}}(\rho) = -\frac{Z^{**}}{\rho} - \frac{1}{2}\alpha^2 \tag{3.52}$$

$$Z^{**} \equiv Z + \alpha. \tag{3.53}$$

Apart from some additive constant, it is therefore similar to $U_3(\rho)$.

The results proved in sections 2.3 and 3.1 lead to the spectrum

$$e_{nl} = -\frac{\left[Z - \frac{\alpha}{2}(L^2-1)\right]^2}{2n^2} - \frac{\alpha^2}{8}n^2 + \frac{\alpha}{2}\left[Z + \frac{\alpha}{2}(L^2+1)\right] \tag{3.54}$$

where $n = k + l + 1$ and the allowed k and l values are determined by the inequality

$$(l+k+1)^2 + l(l+1) < \frac{2Z}{\alpha} + 1. \tag{3.55}$$

The corresponding wavefunctions $\psi_{klm}(\rho, \theta, \varphi)$ are given by equations (3.8) and (3.32) with Z^{**} substituted for Z.

4. Conclusion

In the present paper, we have shown that there exist some intimate connections between three unconventional Schrödinger equations based on the use of some deformed canonical commutation relations, of a position-dependent effective mass or of a curved space, respectively. This occurs whenever a specific relation between the deforming function $f(\boldsymbol{x})$, the position-dependent mass $m(\boldsymbol{x})$ and the (diagonal) metric tensor $g(\boldsymbol{x})$ holds true (see equation (2.44)).

As a consequence, any ES Schrödinger equation known in one of these three fields can be reinterpreted as an ES Schrödinger equation in the other two. For instance, the resolution of the three-dimensional harmonic oscillator problem with nonzero minimal uncertainty in position, carried out in [25, 26], provides us, after interchanging the roles of x_i and p_i, with the solution of the three-dimensional harmonic oscillator problem for a position dependence of the mass given by $m(r) = m_0/(1+\overline{\alpha}r^2)^2$, where $\overline{\alpha} \geqslant 0$.

Here we have given another illustration of such a type of relations by considering the Coulomb potential $V(r) = -Ze^2/r$ for a deforming function $f(r) = 1 + \overline{\alpha}r$ ($\overline{\alpha} \geqslant 0$) or a position-dependent mass $m(r) = m_0/(1+\overline{\alpha}r)^2$ or else a similar potential (then distinct from Coulomb) for a diagonal metric tensor $g(r) = 1/(1+\overline{\alpha}r)^2$. In all the cases, we have derived the bound-state energy spectrum and the corresponding wavefunctions. We have shown that in contrast with the standard case, but in analogy with the Coulomb potential in a space of constant negative curvature [27], there are only a finite number of bound states.

It should be stressed that contrary to many constructions of ES Schrödinger equations with position-dependent mass, which start from some known ES problem with constant mass and then deform the potential while leaving the spectrum unchanged, in our approach we consider a known ES potential and determine the effect of a mass position dependence on the spectrum and wavefunctions.

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Appendix. Space curvature and Coulomb potential in curved space

The curvature of the space, whose squared line element is given by equation (2.28), can be expressed as [37]

$$R = g^{ik} R_{ik} \tag{A.1}$$

in terms of the inverse of the metric tensor g^{ik} and the (contracted) curvature tensor

$$R_{ik} = \frac{\partial \Gamma^l_{il}}{\partial X^k} - \frac{\partial \Gamma^l_{ik}}{\partial X^l} + \Gamma^r_{il}\Gamma^l_{kr} - \Gamma^r_{ik}\Gamma^l_{lr} \tag{A.2}$$

where

$$\Gamma^i_{kl} = g^{im}\Gamma_{m,kl} \tag{A.3}$$

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial X^l} + \frac{\partial g_{il}}{\partial X^k} - \frac{\partial g_{kl}}{\partial X^i}\right). \tag{A.4}$$

In this appendix, we restrict ourselves to a diagonal metric tensor depending only on $\rho = \left(\sum_i X_i^2\right)^{1/2}$, i.e., $g_{ij} = \delta_{i,j} g(\rho) = \delta_{i,j}/f^2(\rho)$. Then g^{ij} is given by $g^{ij} = \delta_{i,j}/g(\rho)$ and a straightforward calculation leads us to

$$\Gamma^i_{kl} = a(\rho)(\delta_{i,k}X^l + \delta_{i,l}X^k - \delta_{k,l}X^i) \tag{A.5}$$

$$R_{ik} = \left(\frac{a'(\rho)}{\rho} - a^2(\rho)\right) X^i X^k + \delta_{i,k}[4a(\rho) + \rho a'(\rho) + \rho^2 a^2(\rho)] \tag{A.6}$$

where the prime denotes derivative with respect to ρ and

$$a(\rho) = \frac{g'(\rho)}{2\rho g(\rho)}. \tag{A.7}$$

From (A.1) and (A.6), we obtain

$$R = \frac{2}{g(\rho)}\{4a(\rho) + 2[\rho a(\rho)]' + \rho^2 a^2(\rho)\} \tag{A.8}$$

which can also be rewritten in terms of $f(\rho)$ as shown in equation (2.43).

The Coulomb potential $\phi(\rho)$ in such a curved space can be obtained as a solution of the Laplace equation

$$\Delta\phi(\rho) = 0 \tag{A.9}$$

going to q/ρ when $f(\rho) \to 1$ (with q being the electric charge).

From equation (2.34), it follows that we can rewrite (A.9) as

$$f^2(\rho)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} - \frac{f'(\rho)}{\rho f(\rho)}\boldsymbol{X}\cdot\boldsymbol{\nabla}\right)\phi(\rho) = 0 \tag{A.10}$$

or

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+\frac{2}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}-\frac{f'(\rho)}{f(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\right)\phi(\rho)=0. \tag{A.11}$$

The (nonconstant) solution of this equation reads

$$\phi(\rho)=C_1\int^{\rho}\mathrm{d}\rho'\frac{f(\rho')}{\rho'^2}+C_2 \tag{A.12}$$

where C_1 and C_2 are two integration constants. For $f(\rho)=1$, we recover $\phi(\rho)=q/\rho$ by setting $C_1=-q$ and $C_2=0$. Hence we may choose

$$\phi(\rho)=-q\int^{\rho}\mathrm{d}\rho'\frac{f(\rho')}{\rho'^2}. \tag{A.13}$$

In the special case $f(\rho)=1+\alpha\rho$ considered in section 3.3, the Coulomb potential is therefore given by

$$\phi(\rho)=\frac{q}{\rho}-q\alpha\ln\rho. \tag{A.14}$$

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